Example: Expressing Vector Fields with Coordinate Systems

Consider the vector **field**:

$$\mathbf{A} = xz\,\hat{a}_x + \left(x^2 + y^2\right)\hat{a}_y + \left(\frac{x}{z}\right)\hat{a}_z$$

Let's try to accomplish **three** things:

1. Express **A** using **spherical** coordinates and **Cartesian** base vectors.

2. Express **A** using **Cartesian** coordinates and **spherical** base vectors.

3. Express **A** using **cylindrical** coordinates and **cylindrical** base vectors.

1. The vector field is **already** expressed with Cartesian **base vectors**, therefore we only need to change the Cartesian **coordinates** in each **scalar component** into spherical coordinates.

The **scalar component** of **A** in the *x*-direction is:

$$\begin{aligned} A_x &= xz \\ &= (r\sin\theta\cos\phi)(r\cos\theta) \\ &= r^2\sin\theta\cos\theta\cos\phi \end{aligned}$$

The scalar component of **A** in the *y*-direction is:

$$\begin{aligned} A_y &= x^2 + y^2 \\ &= (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 \\ &= r^2\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) \\ &= r^2\sin^2\theta \end{aligned}$$

The scalar component of **A** in the *z*-direction is:

$$\begin{aligned} A_z &= \frac{x}{z} \\ &= \frac{r\sin\theta\cos\phi}{r\cos\theta} \\ &= \tan\theta\cos\phi \end{aligned}$$

Therefore, the vector field can be expressed using *spherical **coordinates*** as:

$$\mathbf{A} = r^2\sin\theta\cos\theta\cos\phi\,\hat{a}_x + r^2\sin^2\theta\,\hat{a}_y + \tan\theta\cos\phi\,\hat{a}_z$$

2. Now, let's express **A** using **spherical base vectors**. We **cannot** simply change the **coordinates** of each component. Rather, we must determine **new scalar components**, since we are using a **new** set of **base vectors**. We begin by stating:

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{a}_r\right)\hat{a}_r + \left(\mathbf{A}\cdot\hat{a}_\theta\right)\hat{a}_\theta + \left(\mathbf{A}\cdot\hat{a}_\phi\right)\hat{a}_\phi$$

The scalar component A_r is therefore:

$$\begin{aligned}
\mathbf{A}\cdot\hat{a}_r &= xz\,\hat{a}_x\cdot\hat{a}_r + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_r + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_r \\
&= xz\left(\sin\theta\cos\phi\right) + \left(x^2+y^2\right)\left(\sin\theta\sin\phi\right) + \left(\frac{x}{z}\right)\left(\cos\theta\right) \\
&= xz\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}} \\
&\quad + \left(x^2+y^2\right)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}} \\
&\quad + \left(\frac{x}{z}\right)\frac{z}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{x^2 z}{\sqrt{x^2+y^2+z^2}} + \frac{y\left(x^2+y^2\right)}{\sqrt{x^2+y^2+z^2}} + \frac{x}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

Likewise, the scalar component A_θ is:

$$\mathbf{A}\cdot\hat{a}_\theta = xz\,\hat{a}_x\cdot\hat{a}_\theta + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\theta + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\theta$$

$$= xz\left(\cos\theta\cos\phi\right) + \left(x^2+y^2\right)\left(\cos\theta\sin\phi\right) - \left(\frac{x}{z}\right)\left(\sin\theta\right)$$

$$= xz\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}}$$

$$+\left(x^2+y^2\right)\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}}$$

$$-\left(\frac{x}{z}\right)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x^2z^3}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} + \frac{yz^2\left(x^2+y^2\right)}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

$$-\frac{x\left(x^2+y^2\right)}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

$$= \frac{x^2z^3+x^2yz^2+y^3z-x^3-xy^2}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

And finally, the scalar component A_ϕ is:

$$\mathbf{A}\cdot\hat{a}_\phi = xz\,\hat{a}_x\cdot\hat{a}_\phi + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\phi + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\phi$$

$$= xz\left(-\sin\phi\right) + \left(x^2+y^2\right)\left(\cos\phi\right) + \left(\frac{x}{z}\right)0$$

$$= xz\frac{-y}{\sqrt{x^2+y^2}} + \left(x^2+y^2\right)\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{-xyz+x^3+xy^2}{\sqrt{x^2+y^2}}$$

Whew! We're finished! The vector **A** is expressed using Cartesian coordinates and **spherical** base vectors as:

$$\mathbf{A} = \left(\frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{a}_r$$

$$+ \left(\frac{x^2 z^3 + x^2 y z^2 + y^3 z - x^3 - x y^2}{z \sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \right) \hat{a}_\theta$$

$$+ \left(\frac{-xyz + x^3 + xy^2}{\sqrt{x^2 + y^2}} \right) \hat{a}_\phi$$

3. Now, let's write **A** in terms of cylindrical coordinates **and** cylindrical base vectors (i.e., in terms of the cylindrical coordinate **system**).

$$\mathbf{A} = \left(\mathbf{A} \cdot \hat{a}_\rho \right) \hat{a}_\rho + \left(\mathbf{A} \cdot \hat{a}_\phi \right) \hat{a}_\phi + \left(\mathbf{A} \cdot \hat{a}_z \right) \hat{a}_z$$

First, A_ρ is:

$$\begin{aligned} \mathbf{A} \cdot \hat{a}_\rho &= xz\, \hat{a}_x \cdot \hat{a}_\rho + \left(x^2 + y^2 \right) \hat{a}_y \cdot \hat{a}_\rho + \left(\frac{x}{z} \right) \hat{a}_z \cdot \hat{a}_\rho \\ &= xz \left(\cos\phi \right) + \left(x^2 + y^2 \right) \left(\sin\phi \right) + \left(\frac{x}{z} \right) (0) \\ &= \rho \cos\phi\, z \left(\cos\phi \right) + \rho^2 \left(\sin\phi \right) \\ &= \rho \cos^2\phi\, z + \rho^2 \sin\phi \end{aligned}$$

And A_ϕ is:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_\phi &= xz\,\hat{a}_x\cdot\hat{a}_\phi + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\phi + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\phi \\ &= xz\left(-\sin\phi\right) + \left(x^2+y^2\right)\left(\cos\phi\right) + \left(\frac{x}{z}\right)(0) \\ &= -\rho\cos\phi\, z\left(\sin\phi\right) + \rho^2\left(\cos\phi\right) \\ &= \rho\cos\phi\left(\rho - z\sin\phi\right)\end{aligned}$$

And finally, A_z is:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_z &= xz\,\hat{a}_x\cdot\hat{a}_z + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_z + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_z \\ &= xz(0) + \left(x^2+y^2\right)(0) + \left(\frac{x}{z}\right)(1) \\ &= \left(\frac{x}{z}\right) \\ &= \frac{\rho\cos\phi}{z}\end{aligned}$$

We can therefore express the vector field **A** using **both** cylindrical coordinates and cylindrical base vectors:

$$\mathbf{A} = \left(\rho\cos^2\phi\, z + \rho^2\sin\phi\right)\hat{a}_\rho + \rho\cos\phi\left(\rho - z\sin\phi\right)\hat{a}_\phi + \left(\frac{\rho\cos\phi}{z}\right)\hat{a}_z$$

Thus, we have determined **three** possible ways (and there are many other ways!) to express the vector field **A**:

1.

$$\mathbf{A} = r^2 \sin\theta \, \cos\theta \, \cos\phi \, \hat{a}_x + r^2 \sin^2\theta \, \hat{a}_y + \tan\theta \cos\phi \, \hat{a}_z$$

2.

$$\begin{aligned}\mathbf{A} = &\left(\frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{a}_r \\ &+ \left(\frac{x^2 z^3 + x^2 y z^2 + y^3 z - x^3 - x y^2}{z \sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \right) \hat{a}_\theta \\ &+ \left(\frac{-xyz + x^3 + x y^2}{\sqrt{x^2 + y^2}} \right) \hat{a}_\phi\end{aligned}$$

3.

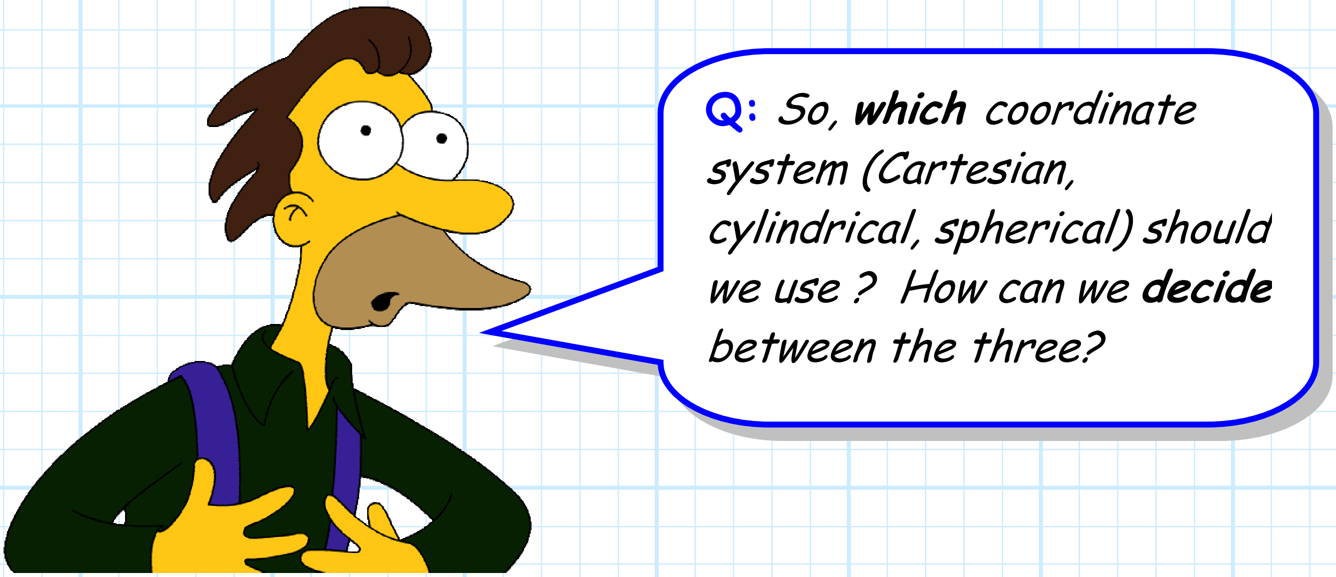
$$\mathbf{A} = \left(\rho \cos^2\phi \, z + \rho^2 \sin\phi \right) \hat{a}_\rho + \rho \cos\phi \left(\rho - z \sin\phi \right) \hat{a}_\phi + \left(\frac{\rho \cos\phi}{z} \right) \hat{a}_z$$

Please note:

* The three expressions for vector field **A** provided in this handout each **look** very different. However, they are just three different methods for describing the **same** vector field. **Any** one of the three is correct, and will result in the **same result** for any physical problem.

* We can express a vector field using **any** set of coordinate variables **and** any set of base vectors.

* Generally speaking, however, we use one coordinate **system** to describe a vector field. For example, we use **both** spherical coordinates and spherical base vectors.



A: Ideally, we select that system that most **simplifies** the mathematics. This depends on the **physical problem** we are solving.

For example, if we are determining the fields resulting from a **spherically symmetric** charge density, we will find that using the **spherical** coordinate system will make our analysis the easiest and most straightforward.